

# A complement on representations of Hom-Lie algebras

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## Abstract

In this paper, we give a new series of coboundary operators of Hom-Lie algebras. And prove that cohomology groups with respect to coboundary operators are isomorphic. Then, we revisit representations of Hom-Lie algebras, and generalize the relation between Lie algebras and their representations to Hom-Lie algebras.

## 1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [5] as part of a study of deformations of the Witt and the Virasoro algebras. In a hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Some  $q$ -deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [5, 6]. Because of close relation to discrete and deformed vector fields and differential calculus [5, 7, 8], more people pay special attention to this algebraic structure. In particular, Hom-Lie algebras on semisimple Lie algebras are studied in [11]; Its geometric generalization is given in [9]; Quadratic hom-Lie algebras are studied in [2]; Representation theory, cohomology and homology theory are systematically studied in [1, 3, 14, 19]; Bialgebra theory and Hom-(Classical) Yang-Baxter Equation are studied in [13, 15, 18, 23]; The notion of a Hom-Lie 2-algebra, which is a categorification of a Hom-Lie algebra, is introduced in [16], in which the relation with Hom-left-symmetric algebras [12] and symplectic Hom-Lie algebras are studied.

Let  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra,  $V$  be a vector space,  $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  be a representation of  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  on the vector space  $V$  with respect to  $\beta \in \mathfrak{gl}(V)$ . The set of  $k$ -cochains on  $\mathfrak{g}$  with values in  $V$ , which we denote by  $C^k(\mathfrak{g}; V)$ , is the set of skewsymmetric  $k$ -linear maps from  $\mathfrak{g} \times \cdots \times \mathfrak{g}$  ( $k$ -times) to  $V$ :

$$C^k(\mathfrak{g}; V) := \{\eta : \wedge^k \mathfrak{g} \longrightarrow V \text{ is a linear map}\}.$$

For a part of  $C^k(\mathfrak{g}; V)$ , in [20], there are a series of coboundary operators  $\hat{d}^s$ ; in [21], authors give a especial coboundary operator of regular Hom-Lie algebras. For regular Hom-Lie algebras, there are many works were done by the especial coboundary operator [21, 22]. In this article, we first study representation of Hom-Lie algebras, we give a new series coboundary operators on  $k$ -cochains, prove that Cohomology groups with respect to these coboundary operators, are isomorphic. Then, we revisit representations of Hom-Lie algebras, and generalize the result " If  $\mathfrak{g}$

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is a Lie algebra,  $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  is a representation if and only if there is a degree-1 operator  $D$  on  $\Lambda \mathfrak{g}^* \otimes V$  satisfying  $D^2 = 0$ , and

$$D(\xi \wedge \eta \otimes u) = d_{\mathfrak{g}} \xi \wedge \eta \otimes u + (-1)^k \xi \wedge D(\eta \otimes u), \quad \forall \xi \in \wedge^k \mathfrak{g}^*, \eta \in \wedge^l \mathfrak{g}^*, u \in V,$$

where  $d_{\mathfrak{g}} : \wedge^k \mathfrak{g}^* \longrightarrow \wedge^{k+1} \mathfrak{g}^*$  is the coboundary operator associated to the trivial representation."

The paper is organized as follows. In Section 2, we recall some necessary background knowledge: Hom-Lie algebras and their representations and so on. In Section 3, we show that  $d^s$  is coboundary operators of Hom-Lie algebras and prove that Cohomology groups with respect to these coboundary operators, are isomorphic(Theorem3.4). In Section 4, we give some properties of  $d^s$  and revisit representations of Hom-Lie algebras, and have Theorem4.4.

## 2 Preliminaries

The notion of a Hom-Lie algebra was introduced in [5], see also [2, 12] for more information.

**Definition 2.1.** (1) A Hom-Lie algebra is a triple  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  consisting of a vector space  $\mathfrak{g}$ , a skewsymmetric bilinear map (bracket)  $[\cdot, \cdot] : \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}$  and a linear transformation  $\alpha : \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying  $\alpha[x, y] = [\alpha(x), \alpha(y)]$ , and the following Hom-Jacobi identity:

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \quad \forall x, y, z \in \mathfrak{g}. \quad (1)$$

A Hom-Lie algebra is called a regular Hom-Lie algebra if  $\alpha$  is a linear automorphism.

- (2) A subspace  $\mathfrak{h} \subset \mathfrak{g}$  is a Hom-Lie sub-algebra of  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  if  $\alpha(\mathfrak{h}) \subset \mathfrak{h}$  and  $\mathfrak{h}$  is closed under the bracket operation  $[\cdot, \cdot]$ , i.e. for all  $x, y \in \mathfrak{h}$ ,  $[x, y] \in \mathfrak{h}$ .
- (3) A morphism from the Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, \alpha)$  to the hom-Lie algebra  $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, \gamma)$  is a linear map  $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$  such that  $\psi([x, y]_{\mathfrak{g}}) = [\psi(x), \psi(y)]_{\mathfrak{h}}$  and  $\psi \circ \alpha = \gamma \circ \psi$ .

Representation and cohomology theories of Hom-Lie algebra are systematically introduced in [1, 14]. See [19] for homology theories of Hom-Lie algebras.

**Definition 2.2.** A representation of the Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  on a vector space  $V$  with respect to  $\beta \in \mathfrak{gl}(V)$  is a linear map  $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ , such that for all  $x, y \in \mathfrak{g}$ , the following equalities are satisfied:

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x); \quad (2)$$

$$\rho([x, y]) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x). \quad (3)$$

## 3 Cohomology operators of Hom-Lie algebras

Let  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  be a Hom-Lie algebra,  $V$  be a vector space,  $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  be a representation of  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  on the vector space  $V$  with respect to  $\beta \in GL(V)$ , where  $\beta$  is reversible. In this paper, we just consider  $\beta \in GL(V)$ .

The set of  $k$ -cochains on  $\mathfrak{g}$  with values in  $V$ , which we denote by  $C^k(\mathfrak{g}; V)$ , is the set of skewsymmetric  $k$ -linear maps from  $\mathfrak{g} \times \cdots \times \mathfrak{g}$  ( $k$ -times) to  $V$ :

$$C^k(\mathfrak{g}; V) := \{\eta : \wedge^k \mathfrak{g} \longrightarrow V \text{ is a linear map}\}.$$

For  $s = 0, 1, 2, \dots$ , define  $d^s : C^k(\mathfrak{g}; V) \longrightarrow C^{k+1}(\mathfrak{g}; V)$  by

$$\begin{aligned} d^s \eta(x_1, \dots, x_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \beta^{k+1+s} \rho(x_i) \beta^{-k-2-s} \eta(\alpha(x_1), \dots, \widehat{x_i}, \dots, \alpha(x_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{k+1})), \end{aligned}$$

where  $\beta^{-1}$  is the inverse of  $\beta$ ,  $\eta \in C^k(\mathfrak{g}; V)$ .

**Proposition 3.1.** *With the above notations, the map  $d^s$  is a coboundary operator, i.e.  $d^s \circ d^s = 0$ .*

**Proof.** For any  $\eta \in C^k(\mathfrak{g}; V)$ , by straightforward computations, we have

$$\begin{aligned} d^s \circ d^s \eta(x_1, \dots, x_{k+2}) &= \sum_{i=1}^{k+2} (-1)^{i+1} \beta^{k+2+s} \rho(x_i) \beta^{-k-3-s} d^s \eta(\alpha(x_1), \dots, \widehat{x_i}, \dots, \alpha(x_{k+2})) \\ &\quad + \sum_{i < j} (-1)^{i+j} d^s \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{k+2})). \end{aligned}$$

And

$$\begin{aligned} d^s \eta(\alpha(x_1), \dots, \widehat{x_i}, \dots, \alpha(x_{k+2})) &= \sum_{l < i} (-1)^{l+1} \beta^{k+1+s} \rho(\alpha(x_l)) \beta^{-k-2-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{l,i}}, \dots, \alpha^2(x_{k+2})) \\ &\quad + \sum_{l > i} (-1)^l \beta^{k+1+s} \rho(\alpha(x_l)) \beta^{-k-2-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{i,l}}, \dots, \alpha^2(x_{k+2})) \\ &\quad + \sum_{m < n < i} (-1)^{m+n} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{m,n,i}}, \dots, \alpha^2(x_{k+2})) \\ &\quad + \sum_{m < i < n} (-1)^{m+n-1} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{m,i,n}}, \dots, \alpha^2(x_{k+2})) \\ &\quad + \sum_{i < m < n} (-1)^{m+n} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{i,m,n}}, \dots, \alpha^2(x_{k+2})). \end{aligned}$$

At the same time, we have

$$\begin{aligned}
& d^s \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{k+2})) \\
&= \beta^{k+1+s} \rho([x_i, x_j]) \beta^{-k-2-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha^2(x_{k+2})) \\
&+ \sum_{p < i < j} (-1)^p \beta^{k+1+s} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{p,i,j}}, \dots, \alpha^2(x_{k+2})) \\
&+ \sum_{i < p < j} (-1)^{p+1} \beta^{k+1+s} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{i,p,j}}, \dots, \alpha^2(x_{k+2})) \\
&+ \sum_{i < j < p} (-1)^p \beta^{k+1+s} \rho(\alpha(x_p)) \beta^{-k-2-s} \eta(\alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{i,j,p}}, \dots, \alpha^2(x_{k+2})) \\
&+ \sum_{q < i < j} (-1)^{1+q} \eta([[x_i, x_j], \alpha(x_q)], \alpha^2(x_1), \dots, \widehat{x_{q,i,j}}, \dots, \alpha^2(x_{k+2})) \tag{4} \\
&+ \sum_{i < q < j} (-1)^q \eta([[x_i, x_j], \alpha(x_q)], \alpha^2(x_1), \dots, \widehat{x_{i,q,j}}, \dots, \alpha^2(x_{k+2})) \tag{5} \\
&+ \sum_{i < j < q} (-1)^{1+q} \eta([[x_i, x_j], \alpha(x_q)], \alpha^2(x_1), \dots, \widehat{x_{i,j,q}}, \dots, \alpha^2(x_{k+2})) \tag{6} \\
&+ \sum_{m < n < i < j} (-1)^{m+n} \eta([\alpha(x_m), \alpha(x_n)], \alpha([x_i, x_j]), \alpha^2(x_1), \dots, \widehat{x_{m,n,i,j}}, \dots, \alpha^2(x_{k+2})) \tag{7} \\
&+ \dots
\end{aligned}$$

By Hom-Jacobi identity:

$$(4) + (5) + (6) = 0,$$

and we have:  $(7) + \dots = 0$ .

By  $\rho(\alpha(x))\beta = \beta\rho(x)$  and  $\rho(\alpha(x)) = \beta\rho(x)\beta^{-1}$ , we have  $d^s \circ d^s \eta(x_1, \dots, x_{k+2})$

$$= \sum_{l < i} (-1)^{l+i} \beta^{k-1+s} \rho(\alpha^3(x_i)) \rho(\alpha^2(x_l)) \beta^{-k-1-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{l,i}}, \dots, \alpha^2(x_{k+2})) \tag{8}$$

$$+ \sum_{l > i} (-1)^{l+i+1} \beta^{k-1+s} \rho(\alpha^3(x_i)) \rho(\alpha^2(x_l)) \beta^{-k-1-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{i,l}}, \dots, \alpha^2(x_{k+2})) \tag{9}$$

$$+ \sum_{m < n < i} (-1)^{m+n+i+1} \beta^{k+2+s} \rho(x_i) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{m,n,i}}, \dots, \alpha^2(x_{k+2}))$$

$$+ \sum_{m < i < n} (-1)^{m+n+i} \beta^{k+2+s} \rho(x_i) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{m,i,n}}, \dots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i < m < n} (-1)^{m+n+i+1} \beta^{k+2+s} \rho(x_i) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{i,m,n}}, \dots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i < j} (-1)^{i+j} \beta^{k-1+s} \rho([\alpha^2(x_i), \alpha^2(x_j)]) \beta^{-k-1-s} \eta(\alpha^2(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha^2(x_{k+2})) \tag{10}$$

$$+ \sum_{p < i < j} (-1)^{p+i+j} \beta^{k+2+s} \rho(x_p) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{p,i,j}}, \dots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i < p < j} (-1)^{p+i+j+1} \beta^{k+2+s} \rho(x_p) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{i,p,j}}, \dots, \alpha^2(x_{k+2}))$$

$$+ \sum_{i < j < p} (-1)^{p+i+j} \beta^{k+2+s} \rho(x_p) \beta^{-k-3-s} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{i,j,p}}, \dots, \alpha^2(x_{k+2})).$$

By  $\rho([x, y])\beta = \rho(\alpha(x))\rho(y) - \rho(\alpha(y))\rho(x)$ , we have

$$(8) + (9) + (10) = 0.$$

About above equations, sum of the rest six equations is zero. So, we proof that  $d^s \circ d^s = 0$ . ■

**Remark 3.2.** In [20], coboundary operators  $\hat{d}^s : C_{\alpha, \beta}^k(\mathfrak{g}; V) \longrightarrow C_{\alpha, \beta}^{k+1}(\mathfrak{g}; V)$ , where  $C_{\alpha, \beta}^k(\mathfrak{g}; V) = \{\eta \in C^k(\mathfrak{g}; V) | \eta \circ \alpha = \beta \circ \eta\}$ . The coboundary operators  $d^s$  we define are not the same as those in [21], [21] just give coboundary operator for regular Hom-Lie algebras.

From  $d^s$ , we know the coboundary operator associated to the trivial representation is  $d : \wedge^k \mathfrak{g}^* \longrightarrow \wedge^{k+1} \mathfrak{g}^*$ ,

$$d\xi(x_1, \dots, x_{k+1}) = \sum_{i < j} (-1)^{i+j} \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{k+1})).$$

For Hom-Lie algebra  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  and representation  $\rho$  with respect with  $\beta$ ,  $\alpha$  induces a map  $\bar{\alpha} : C^l(\mathfrak{g}; V) \longrightarrow C^l(\mathfrak{g}; V)$  via

$$\bar{\alpha}(\eta)(x_1, \dots, x_l) = \eta(\alpha(x_1), \dots, \alpha(x_l)).$$

And  $\beta$  induces a map  $\bar{\beta} : C^l(\mathfrak{g}; V) \longrightarrow C^l(\mathfrak{g}; V)$  via

$$\bar{\beta}(\eta)(x_1, \dots, x_l) = \beta \circ \eta(x_1, \dots, x_l).$$

$C^\bullet(\mathfrak{g}; V) = \oplus_l C^l(\mathfrak{g}; V)$  is a  $\wedge^\bullet = \oplus_k \wedge^k \mathfrak{g}^*$ -module, where the action  $\diamond : \wedge^k \mathfrak{g}^* \times C^l(\mathfrak{g}; V) \longrightarrow C^{k+l}(\mathfrak{g}; V)$  is given by

$$\xi \diamond \eta(x_1, \dots, x_{k+l}) = \sum_{\kappa} \text{sgn}(\kappa) \eta(x_{\kappa(1)}, \dots, x_{\kappa(k)}) \eta(x_{\kappa(k+1)}, \dots, x_{\kappa(k+l)}),$$

where  $\xi \in \wedge^k \mathfrak{g}^*$ ,  $\eta \in C^l(\mathfrak{g}; V)$ , and the summation is taken over  $(k, l)$ -unshuffles.

Obviously, for  $\xi, \xi_1, \xi_2 \in \wedge^\bullet, \eta \in C^\bullet(\mathfrak{g}; V)$ , we have

$$\begin{aligned} \bar{\alpha}(\xi_1 \wedge \xi_2) &= \bar{\alpha}(\xi_1) \wedge \bar{\alpha}(\xi_2); \\ \bar{\alpha}(\xi \diamond \eta) &= \bar{\alpha}(\xi) \diamond \bar{\alpha}(\eta); \\ \bar{\beta}(\xi \diamond \eta) &= \xi \diamond \bar{\beta}(\eta). \end{aligned}$$

Associated to the representation  $\rho$ , we obtain the complex  $(C^k(\mathfrak{g}; V), d^s)$ . Denote the set of closed  $k$ -cochains by  $Z^k(d^s)$  and the set of exact  $k$ -cochains by  $B^k(d^s)$ . Denote the corresponding cohomology by

$$H^k(d^s) = Z^k(d^s) / B^k(d^s).$$

Now, we study the relation between  $H^k(d^s)$  and  $H^k(d^{s+1})$ .

**Proposition 3.3.** *With the above notations, we have*

$$\bar{\beta} \circ d^s = d^{s+1} \circ \bar{\beta}.$$

**Proof.** For  $\eta \in C^l(\mathfrak{g}; V)$ , we have

$$\begin{aligned}
\bar{\beta} \circ d^s \eta(x_1, \dots, x_{l+1}) &= \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+2+s} \rho(x_i) \beta^{-l-2-s} \eta(\alpha(x_1), \dots, \hat{x}_i, \dots, \alpha(x_{l+1})) \\
&\quad + \sum_{i < j} (-1)^{i+j} \beta \circ \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{l+1})) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+2+s} \rho(x_i) \beta^{-l-3-s} \bar{\beta}(\eta)(\alpha(x_1), \dots, \hat{x}_i, \dots, \alpha(x_{l+1})) \\
&\quad + \sum_{i < j} (-1)^{i+j} \bar{\beta}(\eta)([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{l+1})) \\
&= d^{s+1}(\bar{\beta}(\eta))(x_1, \dots, x_{l+1}),
\end{aligned}$$

which implies that  $\bar{\beta} \circ d^s = d^{s+1} \circ \bar{\beta}$ . ■

**Theorem 3.4.** For  $s = 0, 1, 2, \dots$ , we have:  $H^k(d^s) \cong H^k(d^{s+1})$ .

**Proof.** By  $d^{s+1} \circ \bar{\beta} = \bar{\beta} \circ d^s$ , for  $\eta \in Z^k(d^s)$ , we have  $\bar{\beta}(\eta) \in Z^k(d^{s+1})$ . On the other hand, for  $\eta_1 \in B^k(d^s)$ , there is  $\omega \in C^{k-1}(\mathfrak{g}; V)$ , such that:  $\eta_1 = d^s \omega$ . so,

$$\bar{\beta}(\eta_1) = \bar{\beta} \circ d^s \omega = d^{s+1} \circ \bar{\beta}(\omega).$$

Obviously,  $\bar{\beta}(\omega) \in C^{k-1}(\mathfrak{g}; V)$ , then,  $\bar{\beta}(\eta_1) \in B^k(d^{s+1})$ . Actually, we have proof:

$$\bar{\beta}(Z^k(d^s)) \subset Z^k(d^{s+1}), \quad \bar{\beta}(B^k(d^s)) \subset B^k(d^{s+1}).$$

Next, for  $\beta^{-1}$ , we define map  $\overline{\beta^{-1}} : C^k(\mathfrak{g}; V) \rightarrow C^k(\mathfrak{g}; V)$  by

$$\overline{\beta^{-1}}(\eta)(x_1, \dots, x_k) = \beta^{-1} \circ \eta(x_1, \dots, x_k), \quad \forall \eta \in C^k(\mathfrak{g}; V).$$

For  $\eta \in Z^k(d^{s+1})$ , we have  $d^{s+1} \eta = 0$ . By

$$\bar{\beta} \circ d^s \circ \overline{\beta^{-1}}(\eta) = d^{s+1} \circ \bar{\beta} \circ \overline{\beta^{-1}}(\eta) = d^{s+1} \eta = 0.$$

We have:

$$d^s \circ \overline{\beta^{-1}}(\eta) = 0,$$

then, we have:

$$\overline{\beta^{-1}}(\eta) \in Z^k(d^s).$$

On the other hand, for  $\eta_1 \in B^k(d^{s+1})$ , there is  $\omega \in C^{k-1}(\mathfrak{g}; V)$ , such that  $\eta_1 = d^{s+1} \omega$ . Then, we have:

$$\bar{\beta} \circ d^s \circ \overline{\beta^{-1}}(\omega) = d^{s+1} \circ \bar{\beta} \circ \overline{\beta^{-1}}(\omega) = d^{s+1} \omega = \eta_1.$$

So,

$$d^s \circ \overline{\beta^{-1}}(\omega) = \beta^{-1} \circ \eta_1,$$

then

$$\overline{\beta^{-1}}(\eta_1) \in B^k(d^s).$$

Actually, we have proof:

$$\overline{\beta^{-1}}(Z^k(d^{s+1})) \subset Z^k(d^s); \quad \overline{\beta^{-1}}(B^k(d^{s+1})) \subset B^k(d^s).$$

Now, we complete the proof. ■

## 4 Representations of Hom-Lie algebras-revisited

We first consider the coboundary operator  $d$ , which is associated to the trivial representation. The following is right.

**Proposition 4.1.** *For  $\xi_1 \in \wedge^k \mathfrak{g}^*$ ,  $\xi_2 \in \wedge^l \mathfrak{g}^*$ , we have*

$$d(\xi_1 \wedge \xi_2) = d\xi_1 \wedge \bar{\alpha}(\xi_2) + (-1)^k \bar{\alpha}(\xi_1) \wedge d\xi_2.$$

**Proof.** This proof is similar to Proposition 3.2 in [20]. ■

**Proposition 4.2.** *For  $\xi \in \wedge^k \mathfrak{g}^*$ ,  $\eta \in C^l(\mathfrak{g}; V)$ , we have*

$$d^s(\xi \diamond \eta) = d\xi \diamond \bar{\alpha}(\eta) + (-1)^k \bar{\alpha}(\eta) \diamond d^{s+k}\eta.$$

**Proof.** First let  $k = 1$ , then  $\xi \diamond \eta \in C^{l+1}(\mathfrak{g}; V)$ . We have

$$\begin{aligned} & d^s(\xi \diamond \eta)(x_1, \dots, x_{l+2}) \\ &= \sum_{i=1}^{k+2} (-1)^{l+1} \beta^{l+2+s} \rho(x_i) \beta^{-l-3-s} \xi \diamond \eta(x_1, \dots, \widehat{x_i}, \dots, x_{l+2}) \\ & \quad + \sum_{i < j} (-1)^{i+j} \xi \diamond \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{l+2})) \\ &= \sum_{i < j} (-1)^{i+j} \xi([x_i, x_j]) \eta(\alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{l+2})) \\ & \quad + \sum_{q < i} (-1)^{i+q} \xi(\alpha(x_i)) \beta^{l+2+s} \rho(x_q) \beta^{-l-3-s} \eta(\alpha(x_1), \dots, \widehat{x_{q,i}}, \dots, \alpha(x_{l+2})) \\ & \quad + \sum_{i < q} (-1)^{i+q+1} \xi(\alpha(x_i)) \beta^{l+2+s} \rho(x_q) \beta^{-l-3-s} \eta(\alpha(x_1), \dots, \widehat{x_{i,q}}, \dots, \alpha(x_{l+2})) \\ & \quad + \sum_{q < i < j} (-1)^{q+i+j} \xi(\alpha(x_q)) \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{q,i,j}}, \dots, \alpha(x_{l+2})) \\ & \quad + \sum_{i < q < j} (-1)^{q+i+j+1} \xi(\alpha(x_q)) \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,q,j}}, \dots, \alpha(x_{l+2})) \\ & \quad + \sum_{i < j < q} (-1)^{q+i+j} \eta(\alpha(x_q)) \eta([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j,q}}, \dots, \alpha(x_{l+2})) \\ &= d\xi \diamond \bar{\alpha}(\eta)(x_1, \dots, x_{l+2}) + (-1)^1 \bar{\alpha}(\xi) \diamond d^{s+1}\eta(x_1, \dots, x_{l+2}). \end{aligned}$$

Thus, when  $k = 1$ , we have

$$d^s(\xi \diamond \eta) = d\xi \diamond \bar{\alpha}(\eta) + (-1)^1 \bar{\alpha}(\xi) \diamond d^{s+1}\eta.$$

By induction on  $k$ , assume that when  $k = n$ , we have

$$d^s(\xi \diamond \eta) = d\xi \diamond \bar{\alpha}(\eta) + (-1)^n \bar{\alpha}(\xi) \diamond d^{s+n}\eta.$$

For  $\omega \in \mathfrak{g}^*$ ,  $\xi \wedge \omega \in \wedge^{n+1} \mathfrak{g}^*$ , we have

$$\begin{aligned}
d^s((\xi \wedge \omega) \diamond \eta) &= d^s(\xi \diamond (\omega \diamond \eta)) \\
&= d\xi \diamond \bar{\alpha}(\omega \diamond \eta) + (-1)^n \bar{\alpha}(\xi) \diamond d^{s+n}(\omega \diamond \eta) \\
&= (d\xi \wedge \omega) \diamond \bar{\alpha}(\eta) + (-1)^n \bar{\alpha}(\xi) \diamond (d\omega \diamond \bar{\alpha}(\eta) + (-1) \bar{\alpha}(\omega) \diamond d^{s+n+1}\eta) \\
&= (d\xi \wedge \bar{\alpha}(\omega) + (-1)^n \bar{\alpha}(\xi) \wedge d\omega) \diamond \bar{\alpha}(\eta) + (-1)^{n+1} \bar{\alpha}(\xi \wedge \omega) \diamond d^{s+n+1}\eta \\
&= d(\eta \wedge \omega) \diamond \bar{\alpha}(\eta) + (-1)^{n+1} \bar{\alpha}(\eta \wedge \omega) \diamond d^{s+n+1}\eta.
\end{aligned}$$

The proof is completed. ■

**Proposition 4.3.** *With the above notations, we have*

$$\bar{\alpha} \circ d^s = d^{s+1} \circ \bar{\alpha}.$$

**Proof.** For any  $\eta \in C^l(\mathfrak{g}; V)$ , by  $\rho(\alpha(x_i)) = \beta \circ \rho(x_i) \circ \beta$ , we have

$$\begin{aligned}
\bar{\alpha} \circ d^s \eta(x_1, \dots, x_{l+1}) &= d^s \eta(\alpha(x_1), \dots, \alpha(x_{l+1})) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+1+s} \rho(\alpha(x_i)) \beta^{-l-2-s} \eta(\alpha^2(x_1), \dots, \hat{x}_i, \dots, \alpha^2(x_{l+1})) \\
&\quad + \sum_{i < j} (-1)^{i+j} \eta([\alpha(x_i), \alpha(x_j)], \alpha^2(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha^2(x_{l+1})) \\
&= \sum_{i=1}^{l+1} (-1)^{i+1} \beta^{l+2+s} \rho(x_i) \beta^{-l-3-s} \bar{\alpha}(\eta)(\alpha(x_1), \dots, \hat{x}_i, \dots, \alpha(x_{l+1})) \\
&\quad + \sum_{i < j} (-1)^{i+j} \bar{\alpha}(\eta)([x_i, x_j], \alpha(x_1), \dots, \widehat{x_{i,j}}, \dots, \alpha(x_{l+1})) \\
&= d^{s+1}(\bar{\alpha}(\eta))(x_1, \dots, x_{l+1}),
\end{aligned}$$

which implies that  $\bar{\alpha} \circ d^s = d^{s+1} \circ \bar{\alpha}$ . ■

The converse of the above conclusions are also true. Thus, we have the following theorem, which generalize the result " If  $\mathfrak{k}$  is a Lie algebra,  $\rho : \mathfrak{k} \rightarrow \mathfrak{gl}(V)$  is a representation if and only if there is a degree-1 operator  $D$  on  $\Lambda \mathfrak{k}^* \otimes V$  satisfying  $D^2 = 0$ , and

$$D(\xi \wedge \eta \otimes u) = d_{\mathfrak{k}} \xi \wedge \eta \otimes u + (-1)^k \xi \wedge D(\eta \otimes u), \quad \forall \xi \in \wedge^k \mathfrak{k}^*, \eta \in \wedge^l \mathfrak{k}^*, u \in V,$$

where  $d_{\mathfrak{k}} : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$  is the coboundary operator associated to the trivial representation."

**Theorem 4.4.** *Let  $V$  be a vector space,  $\beta \in GL(V)$ . Then  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  is a Hom-Lie algebra, and  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is a representation of  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  on the vector space  $V$  with respect to  $\beta$  if and only if there exists:  $d^s : C^l(\mathfrak{g}; V) \rightarrow C^{l+1}(\mathfrak{g}; V)$ ,  $s = 0, 1, 2, \dots$  and such that:*

- i)  $d^s \circ d^s = 0$ ;
- ii) for any  $\xi \in \wedge^k \mathfrak{g}^*, \eta \in C^l(\mathfrak{g}; V)$ , we have

$$d^s(\xi \diamond \eta) = d\xi \diamond \bar{\alpha}(\eta) + (-1)^k \bar{\alpha}(\xi) \diamond d^{s+k}\eta;$$

where  $d : \wedge^k \mathfrak{g}^* \rightarrow \wedge^{k+1} \mathfrak{g}^*$  is the coboundary operator associated to the trivial representation.



$$\text{iii)} \quad \bar{\alpha} \circ d^s = d^{s+1} \circ \bar{\alpha}.$$

**Proof.** With Propositions we have proof above. We just need to proof the sufficient condition. Sept1, for a fixed map  $\beta \in GL(V)$ . We define  $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$  as follow

$$d^s v(x) = \beta^{1+s} \rho(x) \beta^{-2-s} v, \forall v \in V, x \in \mathfrak{g}. \quad (11)$$

By straightforward computations, we have

$$\begin{aligned} (\bar{\alpha} \circ d^s v(x) = d^s v(\alpha(x)) &= \beta^{1+s} \rho(\alpha(x)) \beta^{-2-s} v, \\ d^{s+1} \circ \bar{\alpha}(v)(x) &= d^{s+1} v(x) = \beta^{2+s} \rho(x) \beta^{-3-s} v. \end{aligned}$$

according to iii), we have:

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x). \quad (12)$$

Sept2, for  $\forall x, y \in \mathfrak{g}, \eta \in C^1(\mathfrak{g}; V)$ , we define  $[\cdot, \cdot] : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$  by

$$\langle \eta, [x, y] \rangle = \beta^{2+s} \rho(x) \beta^{-3-s} \eta(\alpha(y)) - \beta^{2+s} \rho(y) \beta^{-3-s} \eta(\alpha(x)) - d^s \eta(x, y). \quad (13)$$

When  $\xi \in \mathfrak{g}^*$ , according to (13), we have

$$\langle \xi, [x, y] \rangle = -d\xi(x, y). \quad (14)$$

According to (13), (12) and iii), we have

$$\begin{aligned} \langle \eta, [\alpha(x), \alpha(y)] \rangle &= \beta^{2+s} \rho(\alpha(x)) \beta^{-3-s} \eta(\alpha^2(y)) - \beta^{2+s} \rho(\alpha(y)) \beta^{-3-s} \eta(\alpha^2(x)) - d^s \eta(\alpha(x), \alpha(y)) \\ &= \beta^{3+s} \rho(x) \beta^{-4-s} \eta(\alpha^2(y)) - \beta^{3+s} \rho(y) \beta^{-4-s} \eta(\alpha^2(x)) - \bar{\alpha} \circ d^s \eta(x, y) \\ &= \beta^{3+s} \rho(x) \beta^{-4-s} \bar{\alpha}(\eta)(\alpha(y)) - \beta^{3+s} \rho(y) \beta^{-4-s} \bar{\alpha}(\eta)(\alpha(x)) - d^{s+1} \bar{\alpha}(\eta)(x, y) \\ &= \langle \bar{\alpha}(\eta), [x, y] \rangle \\ &= \langle \eta, \alpha([x, y]) \rangle. \end{aligned}$$

So, we have

$$\alpha([x, y]) = [\alpha(x), \alpha(y)]. \quad (15)$$

Sept3, for any  $v \in C^0(\mathfrak{g}; V) = V$ , by i), (13), (11) and (12), we have

$$\begin{aligned} 0 &= d^s \circ d^s v(x, y) \\ &= \beta^{2+s} \rho(x) \beta^{-3-s} d^s v(\alpha(y)) - \beta^{2+s} \rho(y) \beta^{-3-s} d^s v(\alpha(x)) - d^s v([x, y]) \\ &= \beta^{1+s} \rho(\alpha(x)) \rho(y) \beta^{-3-s} v - \beta^{1+s} \rho(\alpha(y)) \rho(x) \beta^{-3-s} v - \beta^{1+s} \rho([x, y]) \beta^{-2-s} v. \end{aligned}$$

We get

$$\rho(\alpha(x)) \rho(y) - \rho(\alpha(y)) \rho(x) = \rho([x, y]) \beta. \quad (16)$$

Sept4, for any  $\xi \in \mathfrak{g}^*, \eta \in C^1(\mathfrak{g}; V)$ , according to ii), (14) and (13), we have

$$\begin{aligned} d^s(\xi \diamond \eta)(x, y, z) &= d\xi \diamond \bar{\alpha}(\eta)(x, y, z) - \bar{\alpha}(\xi) \diamond d^{s+1} \eta(x, y, z) \\ &= d\xi(x, y) \bar{\alpha}(\eta)(z) - d\xi(x, z) \bar{\alpha}(\eta)(y) + d\xi(y, z) \bar{\alpha}(\eta)(x) \\ &\quad - \bar{\alpha}(\xi)(x) d^{s+1} \eta(y, z) + \bar{\alpha}(\xi)(y) d^{s+1} \eta(x, z) - \bar{\alpha}(\xi)(z) d^{s+1} \eta(x, y) \\ &= \beta^{3+s} \rho(x) \beta^{-4-s} (\xi \diamond \eta)(\alpha(y), \alpha(z)) - \beta^{3+s} \rho(y) \beta^{-4-s} (\xi \diamond \eta)(\alpha(x), \alpha(z)) \\ &\quad + \beta^{3+s} \rho(z) \beta^{-4-s} (\xi \diamond \eta)(\alpha(x), \alpha(y)) - \xi \diamond \eta([x, y], \alpha(z)) \\ &\quad + \xi \diamond \eta([x, z], \alpha(y)) - \xi \diamond \eta([y, z], \alpha(x)). \end{aligned}$$

So, for any  $\omega \in C^2(\mathfrak{g}; V)$ , we have

$$\begin{aligned} d^s \omega(x, y, z) &= \beta^{3+s} \rho(x) \beta^{-4-s} \omega(\alpha(y), \alpha(z)) - \beta^{3+s} \rho(y) \beta^{-4-s} \omega(\alpha(x), \alpha(z)) \\ &\quad + \beta^{3+s} \rho(z) \beta^{-4-s} \omega(\alpha(x), \alpha(y)) - \omega([x, y], \alpha(z)) \\ &\quad + \omega([x, z], \alpha(y)) - \omega([y, z], \alpha(x)). \end{aligned}$$

For any  $\eta \in C^1(\mathfrak{g}; V)$ , according to i), we have

$$\begin{aligned} 0 &= d^s \circ d^s \eta(x, y, z) \\ &= \eta([x, y], \alpha(z)) + \eta([y, z], \alpha(x)) + \eta([z, x], \alpha(y)) \end{aligned}$$

Then, we have

$$[[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)] = 0. \quad (17)$$

So, according to (15) and (17), we have:  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  is a Hom-Lie algebra;

according to (12) and (16), we have:  $\rho$  is a representation of  $(\mathfrak{g}, [\cdot, \cdot], \alpha)$  on the vector space  $V$  with respect to  $\beta$ . ■

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